2.4 Vector Multiplication

Vector Multiplication

Key Ideas

- There are two types of vector multiplication: the **dot product** (or **scalar product**) and the **cross product** (or **vector product**).
- The dot product of two vectors results in a scalar quantity.
- The dot product may be evaluated using either the components of the two vectors or using only the magnitudes of the two vectors and the angle between them.
- The dot product has both commutative and distributive properties.
- The cross product of two vectors results in a vector quantity.
- The magnitude and the direction of the vector product may be evaluated either using only the scalar components of the two vectors or using only the magnitudes of the two vectors and the angle between them.
- The cross product has the distributive property, but it does not have the commutative property.
- The vector resulting from the cross product is perpendicular to the plane defined by the input vectors; and the direction may be determined using a right-hand rule.

Learning Objectives

By the end of this section, you will be able to:

- evaluate the scalar product using only the scalar components of the vectors or using the magnitudes of the vectors and the angle between them,
- find the angle between two vectors in a scalar product,
- quickly evaluate the dot product of two Cartesian unit vectors noting that they are either parallel or orthogonal, and
- evaluate the dot product of two vectors, each expressed using Cartesian unit-vector notation.

Previously, we discussed vector addition and subtraction, which resulted in new vectors. However, it is not possible to add a vector and a scalar.

We also discussed the product of a vector and a scalar, which also produces a vector that is either parallel or antiparallel to the original vector.

We will now discuss two types of vector multiplication. The first type, the **dot or scalar product**, results in a scalar, and the second, the **cross or vector product**, results in a vector. A vector can be multiplied by another vector, but it cannot be divided by another vector.

There are a few applications of the dot and cross products that you will encounter in your physics courses. These will be used to define and introduce new physics concepts to you in mechanics (Volume 1) and electricity and magnetism (Volume 2).

The Dot Product of Two Vectors (the Scalar Product)

Scalar multiplication of two vectors yields a scalar product.

The **dot product** (also known as the scalar product) of two vectors, $\vec{a}$ and $\vec{b}$, is defined as

$$\vec{a} \cdot \vec{b} = ab \cos \theta \quad \boxed{2.19}$$

where a and b are the magnitudes of vectors $\vec{a}$ and $\vec{b}$, respectively, and θ is the angle between the two vectors, as shown in the following image.

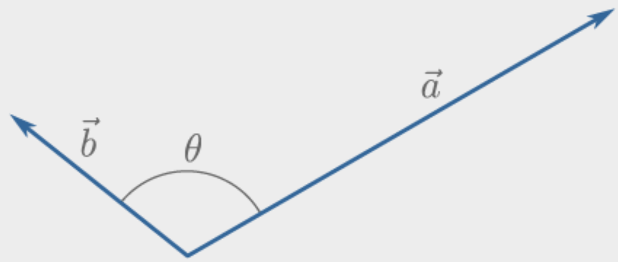

Figure 2.17

In the definition of the dot product, the direction of angle θ is not important. Only the measured value of the angle is important. The dot product of two parallel vectors is $\vec{a} \cdot \vec{b} = ab \cos(0^\circ) = ab$. When two vectors are antiparallel, the dot product is $\vec{a} \cdot \vec{b} = ab \cos(180^\circ) = -ab$.

The dot product of any two vectors that are perpendicular (orthogonal) equals zero because $\vec{a} \cdot \vec{b} = ab \cos(90^\circ) = 0$.

Let's consider a special case, the scalar product of a vector with itself. Noting that every vector is parallel to itself, the angle between the two instances is zero, and hence,

$$\begin{aligned} \vec{a} \cdot \vec{a} &= aa \cos(0^\circ) \\ &= a^2 \\ a &= \sqrt{\vec{a} \cdot \vec{a}} \end{aligned}$$

Consider how this applies to a unit vector. If $\hat{u}$ is a unit vector, then, by definition, it has a magnitude of one, and

$$\hat{u} \cdot \hat{u} = 1$$

Combined with the fact that the scalar product of orthogonal vectors yields zero, the following summary of scalar products between the standard Cartesian unit vectors is very useful.

$$\boxed{2.18}$$

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$$
$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = \hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{i} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = 0$$

Finally, it is useful to explicitly state some properties of the scalar product.

Mathematical Properties of Dot Products

The following mathematical properties apply for the scalar product of any two vectors $\vec{a}$ and $\vec{b}$.

Scalar Multipliers

Multiplying a vector by a scalar (a number) changes the length of the vector. Scalar multipliers factor in the dot product of two vectors with a scalar quantity, s.

$$\vec{a} \cdot (s\vec{b}) = (s\vec{a}) \cdot \vec{b} = s(\vec{a} \cdot \vec{b}) \tag{2.19}$$

Commutative Property

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \tag{2.20}$$

Distributive Property

For any three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$,

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \tag{2.21}$$

Many computations are greatly simplified by using vectors in Cartesian unit-vector notation.

Example 2.10 Dot Product using Cartesian Coordinates

Given the following two vectors, calculate their dot product.

$$\vec{a} = 2\hat{i} - 3\hat{j}$$
$$\vec{b} = 5\hat{j} - 7\hat{k}$$

Strategize

We will use the distributive property, then factor out the scalar multipliers. That will enable us to evaluate the dot products of the Cartesian unit vectors.

Solve

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= \left(2\,\hat{i} - 3\,\hat{j}\right) \cdot \left(5\,\hat{j} - 7\,\hat{k}\right) \\
&= \left(2\,\hat{i}\right) \cdot \left(5\,\hat{j}\right) + \left(2\,\hat{i}\right) \cdot \left(-7\,\hat{k}\right) + \left(-3\,\hat{j}\right) \cdot \left(5\,\hat{j}\right) + \left(-3\,\hat{j}\right) \cdot \left(-7\,\hat{k}\right) \\
&= (2)(5)\left(\hat{i} \cdot \hat{j}\right) - (2)(7)\left(\hat{i} \cdot \hat{k}\right) - (3)(5)\left(\hat{j} \cdot \hat{j}\right) + (3)(7)\left(\hat{j} \cdot \hat{k}\right) \\
&= (2)(5)(0) - (2)(7)(0) - (3)(5)(1) + (3)(7)(0) \\
&= -15
\end{aligned}$$

Assess

As expected, the result is a scalar quantity. With experience, you will learn to drop the terms that evaluate to zero without ever writing them down, and your calculations will become simpler.

We can use the commutative and distributive laws to express the dot product of two vectors in terms of their scalar components.

The Dot Product in Terms of Scalar Components

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z \qquad (2.22)$$